Two-Period Model: State-Preference Approach

"Toutes les généralisations sont dangereuses, y compris celle-ci."
(All generalizations are dangerous, even this one.)
Alexandre Dumas

Relaxing the assumption of mean-variance preferences

Goal of this chapter: relaxing assumption on preferences
Fundamental idea which allows this generalization:

Principle of no arbitrage

It is not possible to get something for nothing.

Basic Two-Period Model

Basic assumptions in Chapter 4

- finite set of investors
- finite set of assets
- finite set of states of the world
- we are taking all of these payoffs into account not only their mean and variance

Representative agent asset Pricing

Idea:

- The price of the asset is equal to the discounted sum of all future payoffs.
- Discount factors are the representative agent's marginal rates of substitution between future consumption and current consumption.

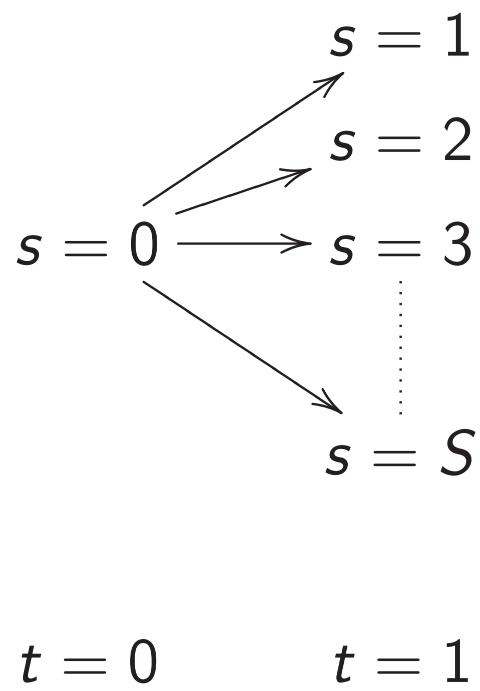
These discount factors are also called the *stochastic discount factors*.
Problem: Assets without payoffs (commodities and hedge funds) have zero price.
$\Rightarrow$ We need to give up the aggregate perspective and look into the trades.

Two-period model

Two periods, $t = 0, 1$:

- $t = 0$ we are in state $s = 0$
- $t = 1$ a finite number of states of the world, $s = 1, 2, \ldots, S$ can occur.

Event tree:



Two-period model

- Assets $k = 0, 1, 2, \ldots, K$.
- first asset, $k = 0$, is the risk free asset
  certain payoff 1 in all second period states
- assets' payoffs denoted by A_s^k.
- price is denoted by q^k
- gross return of asset k in state s is given by $R_s^k := \frac{A_s^k}{q^k}$
- net return is $r_s^k := R_s^k - 1$

Two-period model

Structure of all asset returns in the states-asset-returns-matrix, the SAR-matrix:

$$\boldsymbol{R} := (R_s^k) = \begin{pmatrix} R_1^0 & \cdots & R_1^K \\ \vdots & & \vdots \\ R_S^0 & \cdots & R_S^K \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}^0 & \cdots & \boldsymbol{R}^K \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_1 \\ \vdots \\ \boldsymbol{R}_S \end{pmatrix}.$$

Example: simple way of filling the SAR-matrix with data is to identify each state s with one time period t.

How do we compute mean and covariances of returns from the SAR-matrix?

Mean returns and covariances

Given some probability measure on the set of states, prob_s, we compute

$$\mu(R^k) = \sum_{s=1}^{S} \mathrm{prob}_s R_s^k = \mathbf{prob}' \boldsymbol{R}^k.$$

Covariance matrix

$$COV(\boldsymbol{R}) = \begin{pmatrix} \mathrm{cov}(R^1, R^1) & \cdots & \mathrm{cov}(R^1, R^K) \\ \vdots & & \vdots \\ \mathrm{cov}(R^K, R^1) & \cdots & \mathrm{cov}(R^K, R^K) \end{pmatrix}$$

$$= \boldsymbol{R}' \begin{pmatrix} \mathrm{prob}_1 & & \\ & \ddots & \\ & & \mathrm{prob}_S \end{pmatrix} \boldsymbol{R} - (\boldsymbol{R}'\mathbf{prob})(\mathbf{prob}'\boldsymbol{R}).$$

Factor models (1)

Many factors influence stock returns, e.g.:

- [Chen et al., 1986]
  - Growth rate of industrial production
  - Inflation rate
  - Spread between short-term and long-term interest rates
  - Default risk premia of bonds
- [Mei, 1993]
  - January dummy variable (among other factors)
- [Fama and French, 1993]
  - Premium of a diversified market portfolio
  - Difference between returns of small cap and large cap portfolios
  - Difference between returns of growth and value portfolios

Factor models (2)

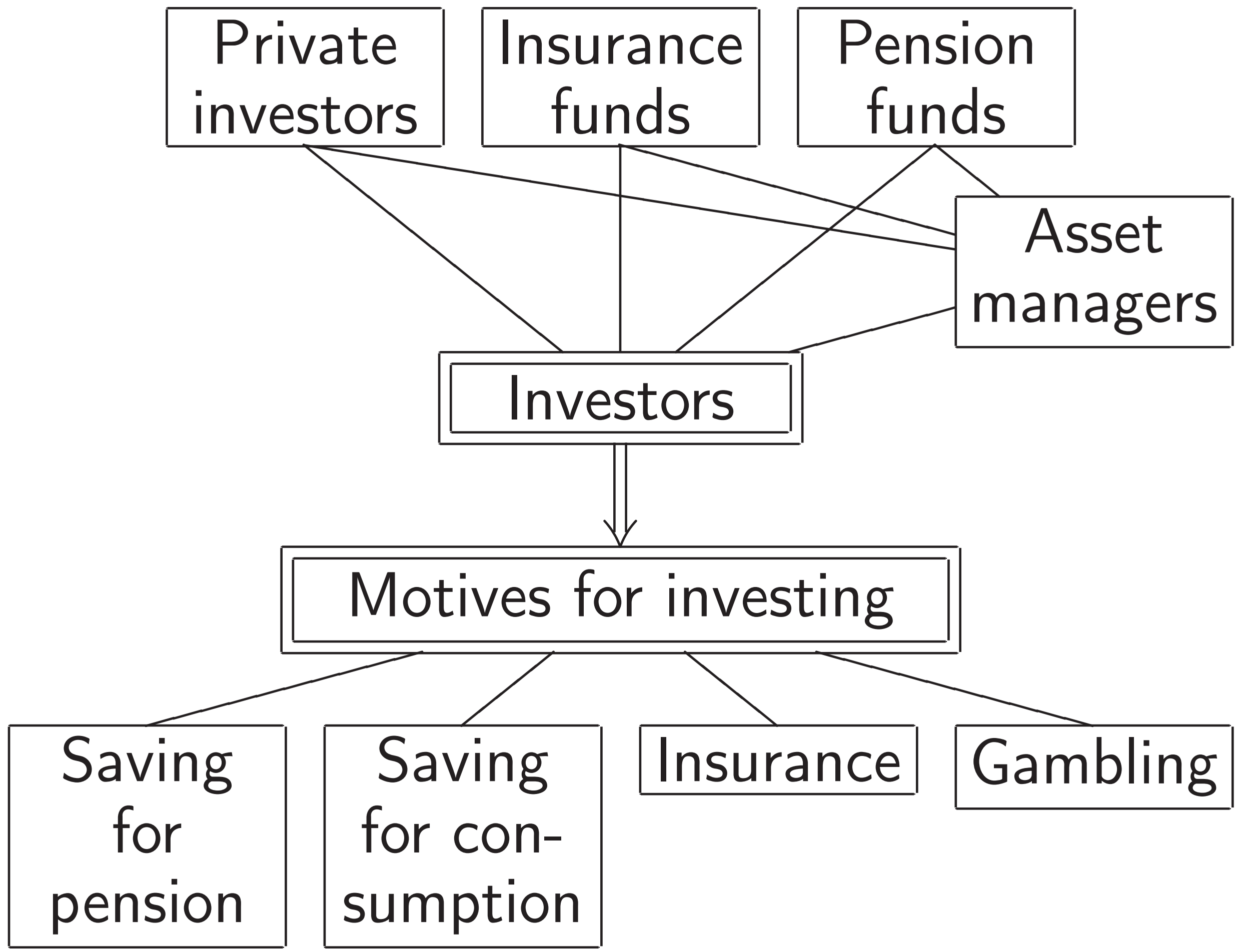
Suppose you can identify $f = 1, \ldots, F$ factors.

- R_s^f = value of factor f in state s.
- β_k^f = sensitivity of asset k's returns to factor f

Then

$$R_s^k = \sum_{f=1}^{F} R_s^f \beta_k^f, \quad \text{i.e.} \quad (R_s^k) = (R_s^f) \cdot (\beta_k^f).$$

Investors – Motives



and other motives

Investors – Model (1)

- Investors $i = 1, \ldots, I$.
- Exogenous wealth $\boldsymbol{w}^i = (w_0^i, w_1^i, \ldots, w_S^i)'$.
- Asset prices $\boldsymbol{q} = (q^0, q^1, \ldots, q^K)'$
- The investors can finance consumption $\boldsymbol{c}^i = (c_0^i, c_1^i, \ldots, c_S^i)'$ by trading the assets.
- $\boldsymbol{\theta}^i = (\theta^{i,0}, \theta^{i,1}, \ldots, \theta^{i,K})'$ vector of asset trade of agent i. $\theta^{i,k}$ can be positive or negative .

Budget restrictions

$$c_0^i + \sum_{k=0}^{K} q^k \theta^{i,k} = w_0^i.$$

If $\sum_{k=0}^{K} q^k \theta^{i,k} < 0$ we say the portfolio is *self-financing*.

Investors – Model (2)

The second period budget constraints are given by:

$$c_s^i = \sum_{k=0}^{K} A_s^k \theta^{i,k} \qquad + w_s^i, \qquad s = 1, \ldots, S.$$

$$\text{consumption} = \text{portfolio value} + \text{exogenous wealth}.$$

An agent wants to maximize consumption c_s^i, but there are obvious limits to how much he can achieve.
How to model this?

There is no "free lunch"

Markets will not offer "free lunches", i.e., arbitrage opportunities (see Sec. 4.2 for a precise definition),
they instead offer trade-offs.

- higher consumption today at the expense of lower consumption tomorrow
- more evenly distributed consumption in all states at the expense of a really high payoff in one of the states.

Preference and trade-offs

- The inter-temporal trade-off is described by time preference discount rate $\delta^i \in (0, 1)$ (Sec. 2.7).
- Preference between states described by von Neumann-Morgenstern utility function (Sec. 2.2)

Both together:

$$U^i(c_0^i, c_1^i, \ldots, c_S^i) = u^i(c_0^i) + \delta^i \sum_{s=1}^{S} \mathrm{prob}_s^i u^i(c_s^i).$$

Assumptions on utility (1)

- If we increase one of the c_s^i, then U^i should also increase. "More money is better, if only for financial reasons."
- We also assume that U is quasi-concave (more evenly distributed consumption is preferred over extreme distributions).

This is the rational way!

Assumptions on utility (2)

General qualitative properties of utility functions:

(1) Continuity: U is continuous on its domain $\mathbb{R}_+^{S+1}$.

(2) Quasi-concavity: the upper contour sets $\{\boldsymbol{c} \in R_+^{S+1} \mid U(\boldsymbol{c}) \geq \mathrm{const}\}$ are convex.

(3) Monotonicity: “More is better”

1. Strict monotonicity: $\boldsymbol{c} > \boldsymbol{c}'$ implies $U(\boldsymbol{c}) > U(\boldsymbol{c}')$.
2. Weak monotonicity: $\boldsymbol{c} \gg \boldsymbol{c}'$ implies $U(\boldsymbol{c}) > U(\boldsymbol{c}')$.

Complete model

We can now summarize the agent's decision problem as:

$$\boldsymbol{\theta}^i = \arg\max_{\boldsymbol{\theta}^i \in R^{K+1}} U^i(\boldsymbol{c}^i) \quad \text{such that} \quad c_0^i + \sum_{k=0}^{K} q^k \theta^{i,k} = w_0^i$$

$$\text{and} \quad c_s^i = \sum_{k=0}^{K} A_s^k \theta^{i,k} + w_s^i, \quad s = 1, \ldots, S.$$

Alternative ways of writing this decision problem can be found in the text book on page 149ff.

Complete and Incomplete Markets

A financial market is

- complete if for all $\boldsymbol{c} \in \mathbb{R}^S$ there exists some $\boldsymbol{\theta} \in \mathbb{R}^{K+1}$ such that $\boldsymbol{c} = \sum_{k=0}^{K} \boldsymbol{A}^k \theta^k$.
- incomplete if some second period consumption streams are not attainable

Complete and Incomplete Markets

Whether financial markets are complete or incomplete depends on the states of the world one is modeling.

- If the states are defined by the assets returns then the market is complete if the variation of the returns is not more frequent than the number of assets.
- If the states are given by exogenous income w then there are insufficient assets to hedge all risks. (Example: students cannot buy securities to insure their future labor income.)

Mathematical condition for completeness

Definition

A market is complete if the rank of the return matrix $\boldsymbol{R}$ is S.

Since $\boldsymbol{R} = \boldsymbol{A}\boldsymbol{\Lambda}(\boldsymbol{q})^{-1}$, the return matrix is complete if and only if the payoff matrix is complete.

Example

Consider

$$\boldsymbol{A}_1 := \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \quad \boldsymbol{A}_2 := \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \boldsymbol{A}_3 := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}.$$

$\boldsymbol{A}_1$ *is complete, but* $\boldsymbol{A}_2$ *and* $\boldsymbol{A}_3$ *are incomplete!*

What Do Agents Trade?

- Agents trade financial assets.
- However, we may also say that agents trade consumption.
- If agents hold heterogeneous beliefs they trade "opinions": they are betting their beliefs.
- Alternative answer: agents trade risk factors.

Hence, whether a financial market model is written in terms of consumption, asset trade or factors is more a matter of convenience.

No-Arbitrage Condition

No-Arbitrage Condition (1)

Suppose

- The shares of Daimler Chrysler are traded at the NYSE for \$90 and in Frankfurt for €70,
- Dollar/Euro exchange rate is 1:1.

What would you do?

Clearly you would buy Daimler Chrysler in Frankfurt and sell it in New York while covering the exchange rate risk by a forward on the Dollar.

Indeed studies show that for double listings differences of less than 1% are erased within 30 seconds. Computer programs immediately exploit this arbitrage opportunity.

No-Arbitrage Condition (2)

Definition

An arbitrage opportunity is a trading strategy that gives you positive returns without requiring any payments.

Arbitrage strategies are so rare one can assume they do not exist.

> *"There is no free lunch" — Milton Friedman*

This simple idea has far reaching conclusions.

Law of one price

Example

Derivatives are assets whose payoffs depend on the payoff of other assets, the underlyings.
Assume the payoff of the derivative can be duplicated by a portfolio of the underlying and a risk free asset. Then the price of the derivative must be the same as the value of the duplicating portfolio.

Generalization:

Law of One Price

The same payoffs need to have the same price.

Implications to restrictions on asset prices

Absence of arbitrage implies restrictions on asset prices:

- Law of One Price requires that asset prices are linear. Doubling all payoffs means doubling the price.
- In mathematical terms, the asset pricing functional is *linear*.

Implications to restrictions on asset prices

Therefore by the Riesz representation theorem (see Appendix A.1, Thm. A.1) there exist weights, called state prices, such that the price of any asset is equal to the weighted sum of its payoffs.

- Absence of arbitrage for mean-variance utilities then implies that the sum of the state prices are positive.
- Absence of arbitrage under weak monotonicity implies that all state prices are non-negative.
- Absence of arbitrage for strictly monotonic utility functions is equivalent to the existence of strictly positive state prices.

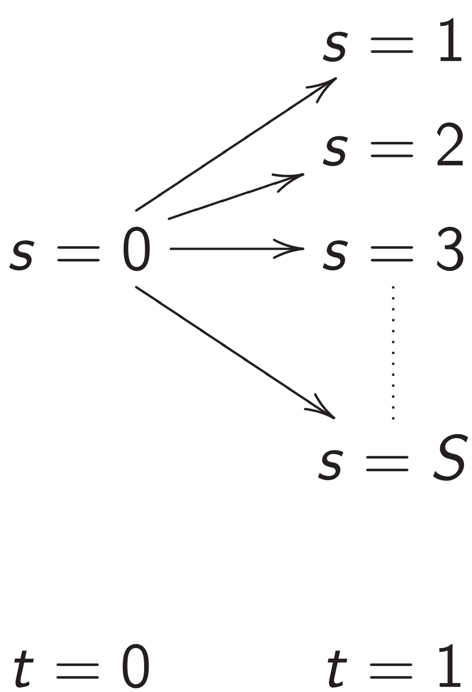
Why different monotonicity assumptions?

We want to build a bridge between

- the economists look at financial markets
- the finance practitioner's point of view, thus we include the case of mean-variance no-arbitrage.

Having understood these two cases you will be able to do the other two cases (Law of One Price and weakly monotonic utilities) easily yourself.

FTAP – Basic model



Two periods, $t = 0, 1$.

- In the second period a finite number of states $s = 1, 2, \ldots, S$ can occur.
- $k = 0, 1, 2, \ldots, K$ assets with payoffs denoted by A_s^k.

States-asset-payoff matrix,

$$\boldsymbol{A} = \begin{pmatrix} A_1^0 & \cdots & A_1^K \\ \vdots & & \vdots \\ A_S^0 & \cdots & A_S^K \end{pmatrix}.$$

Arbitrage (1)

An arbitrage is a trading strategy that an investor would definitely like to exercise.

This definition depends on the investor's utility function.

- For strictly monotonic utility functions an arbitrage is a trading strategy that leads to positive payoffs without requiring any payments.
- For mean-variance utility functions an arbitrage is a trading strategy that offers the risk free payoffs without requiring any payments.

Arbitrage (2)

For strictly monotonic utility functions, an arbitrage is a trading strategy $\boldsymbol{\theta} \in \mathbb{R}^{K+1}$ such that

$$\begin{pmatrix} -\boldsymbol{q}' \\ \boldsymbol{A} \end{pmatrix} \boldsymbol{\theta} > \mathbf{0}.$$

Arbitrage (3)

Example

Payoff matrix is

$$\boldsymbol{A} := \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

while the asset prices are $\boldsymbol{q} = (1, 4)'$. *Can you find an arbitrage opportunity?*

Arbitrage (4)

Solution

Selling one unit of the second asset and buy 3 units of the first asset, you are left with one unit of wealth today, and tomorrow you will be hedged.

How can we erase arbitrage opportunities in this example?

- Obviously asset 2 is too expensive relative to asset 1.

But when is there no arbitrage? We need some mathematics to help us solve this problem!

FTAP

Theorem (Fundamental Theorem of Asset Prices)

The following two statements are equivalent:

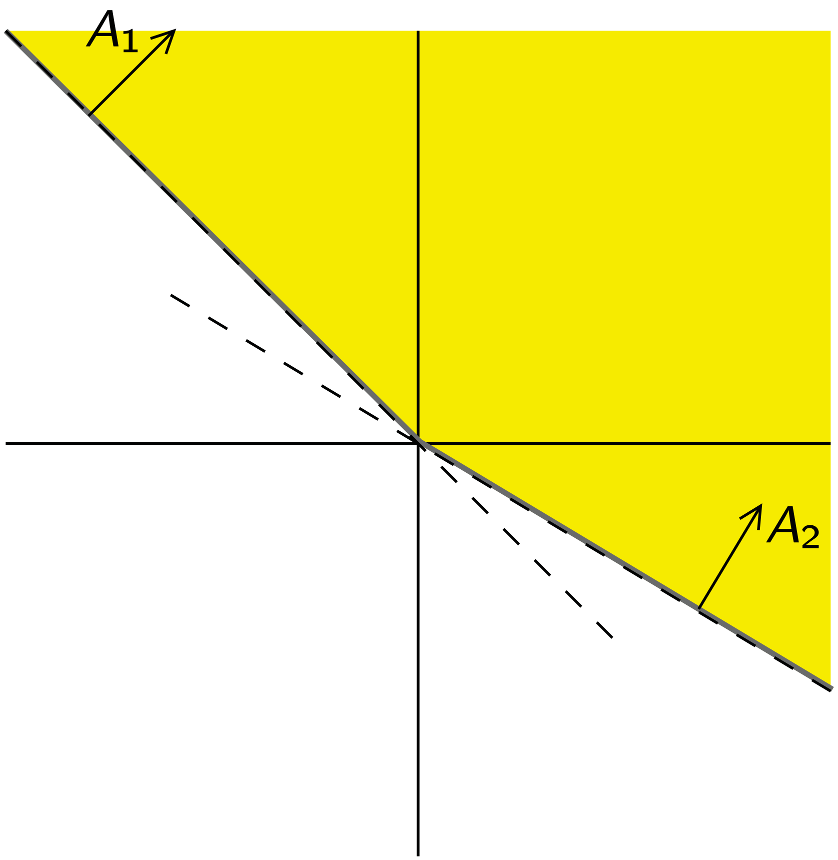
1. *There exists no* $\boldsymbol{\theta} \in \mathbb{R}^{K+1}$ *such that*

$$\begin{pmatrix} -\boldsymbol{q}' \\ \boldsymbol{A} \end{pmatrix} \boldsymbol{\theta} > \mathbf{0}.$$

2. *There exists a* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_s, \ldots, \pi_S)' \in \mathbb{R}^S_{++}$ *such that*

$$q_k = \sum_{s=1}^{S} A_s^k \pi_s, \quad k = 0, \ldots, K.$$

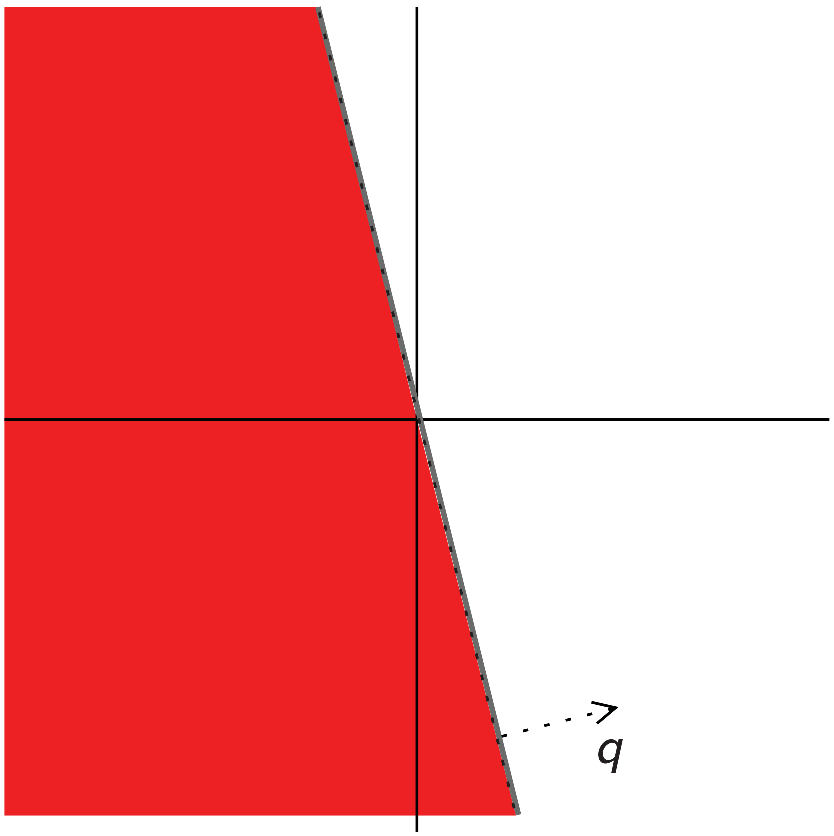
A simple proof for two assets (1)



The case of two assets and two states can be represented by the two dimensional vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$.
First determine

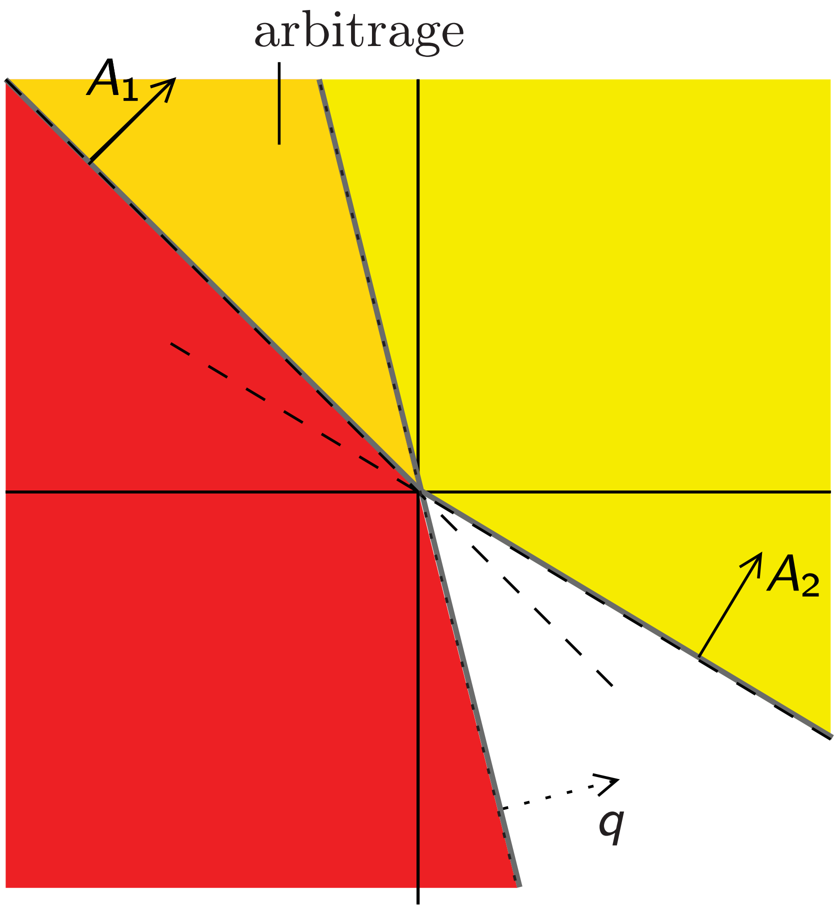
- set of assets where the asset payoff, $\boldsymbol{A}_s\boldsymbol{\theta}$, is equal to 0. This is a line orthogonal to the payoff vector.
- set of non-negative payoffs in both states (yellow).

A simple proof for two assets (2)



Determine the set of strategies requiring no investments, i.e., $-\boldsymbol{q}'\boldsymbol{\theta} \geq 0$ (red).

A simple proof for two assets (3)



Set of arbitrage portfolios is then the intersection of both sets (orange). This set is non-empty if and only if $\boldsymbol{q}$ does not belong to the cone of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, i.e.: if there are no constants $\pi_1, \pi_2 > 0$ such that

$$\boldsymbol{q}' = \pi_1 \boldsymbol{A}_1 + \pi_2 \boldsymbol{A}_2.$$

The proof for the general case can be found in the text book on page 157.

FTAP for mean-variance utility functions

Theorem (FTAP for mean-variance utility functions)

The following two conditions are equivalent:

1. *There exists no* $\boldsymbol{\theta} \in \mathbb{R}^{K+1}$ *such that*

$$\boldsymbol{q}'\boldsymbol{\theta} \leq 0 \quad \textit{and} \quad \boldsymbol{A\theta} = v\mathbf{1}, \quad \textit{for some } v > 0.$$

2. *There exists a* $\boldsymbol{\pi} \in \mathbb{R}^S$ *with* $\sum_{s=1}^S \pi_s > 0$ *such that*

$$q^k = \sum_{s=1}^{S} A_s^k \pi_s, \quad k = 0, \ldots, K.$$

The Proof is analogous to FTAP.

Alternative Formulations of the no-arbitrage principle can be found in the text book on page 158f.

Pricing of Derivatives

The FTAP is essential for the valuation of derivatives.
Two possible ways to determine the value of a derivative:

- determining the value of a *hedge portfolio*.
- use the *risk-neutral probabilities* in order to determine the current value of the derivative's payoff.

Pricing by hedging – example (1)

Example (one-period binomial model)

Current price of a call option on a stock S*. Assume that* $S := 100$ *and there are two possible prices in the next period:* $Su := 200$ *if* $u = 2$ *and* $Sd := 50$ *if* $d = 0.5$.
The riskless interest rate is 10%.
The value of an option with strike price X *is given by* $\max(Su - X, 0)$ *if* u *and* $\max(Sd - X, 0)$ *if* d *is realized.*
We replicate its payoff using the underlying stock and the bond:

- $\max(Su - X, 0) = 200 - 100 = 100$ *in the "up" state,*
- $\max(Sd - X, 0) = \max(50 - 100, 0) = 0$ *in the "down" state.*

Pricing by hedging – example (2)

Example (one-period binomial model (cont.))

The hedge portfolio then requires to borrow $1/3$ *of the risk-free asset and to buy* $2/3$ *risky assets in order to replicate the call's payoff in each of the states:*

$$\begin{aligned} \text{"up":} \quad & \tfrac{2}{3}200 - \tfrac{1}{3}100 = 100 \\ \text{"down":} \quad & \tfrac{2}{3}50 - \tfrac{1}{3}100 = 0 \end{aligned}$$

Pricing by hedging – general case

In general, we need to solve:

$$C_u := \max(Su - X, 0) = nSu + mBR_f$$
$$C_d := \max(Sd - X, 0) = nSd + mBR_f$$

where n is the number of stocks and m is the number of bonds needed to replicate the call payoff.
We get

$$n = \frac{C_u - C_d}{Su - Sd}, \qquad m = \frac{SuC_d - SdC_u}{BR_f(Su - Sd)}$$

The value of the option is therefore:

$$C = nS + mB = \frac{C_u - C_d}{u - d} + \frac{uC_d - dC_u}{R_f(u - d)} = \frac{1}{R_f}\,\frac{C_u(R_f - d) + C_d(u - R_f)}{u - d}.$$

Pricing with state prices (1)

Expected value of the stock with respect to the risk neutral probabilities π^* and $1 - \pi^*$ is

$$S_0 = \pi^* Su + (1 - \pi^*) Sd.$$

This must be the same as investing S today and receiving SR after one period.
Then, $\pi^* Su + (1 - \pi^*) Sd = SR_f$ or $\pi^* u + (1 - \pi^*) d = R_f$. Thus

$$\pi^* = \frac{R_f - d}{u - d}, \quad 0 \leq \pi^* \leq 1.$$

Pricing with state prices (2)

Using the risk-neutral measure we can calculate the current value of the stock and the call:

$$S = \frac{\pi^* Su + (1-\pi^*)Sd}{R_f}, \qquad C = \frac{\pi^* C_u + (1-\pi^*)C_d}{R_f}.$$

Plugging in π^*, we get the price

$$\begin{aligned} C &= \frac{1}{R_f}\left(\frac{R_f - d}{u-d} C_u + \left(1 - \frac{R_f - d}{u-d}\right) C_d\right) \\ &= \frac{1}{R_f}\, \frac{C_u(R_f - d) + C_d(u - R_f)}{u-d}. \end{aligned}$$

Incomplete markets

What about non-redundant derivatives?

- Those can only exist in incomplete markets and applying the Principle of No-Arbitrage will only give valuation bounds.
- For an example see the book (Section 4.2.3). (page 160ff)

Limits to Arbitrage

- In reality investors face *short-sales constraints* and some limits in horizon along which an arbitrage strategy can be carried out.
- The arbitrage is limited and even the law of one price may fail in equilibrium.

Let us first consider an example.

3Com and Palm (1)

On March 2, 2000, 3Com made an IPO of one of its most profitable units. They decided to sell 5% of its Palm stocks and retain 95% thereof.

- At the IPO day, the Palm stock price opened at $38, achieved its high at $165 and closed at $95.06.
- The price of the mother-company 3Com closed that day on $81.81.

3Com and Palm (2)

If we calculate the value of Palm shares per 3Com share (\$142.59), and subtract it from the end price of 3Com, we get

$$\$81.81 - \$142.58 = -\$60.77.$$

Considering the available cash per 3Com share, we would come to a "stub" value for 3Com shares of $-$\$70.77!
This is a contradiction of the law of one price since the portfolio value (negative) differs from the sum of its constituents (positive).

3Com and Palm (3)

- The relative valuation of Palm shares did not open an arbitrage strategy, since it was not possible to short Palm shares.
- Also it was not easy to buy sufficiently many 3Com stocks and then break 3Com apart to sell the embedded Palm stocks.

3Com and Palm (4)

The mismatch persisted for a long time.

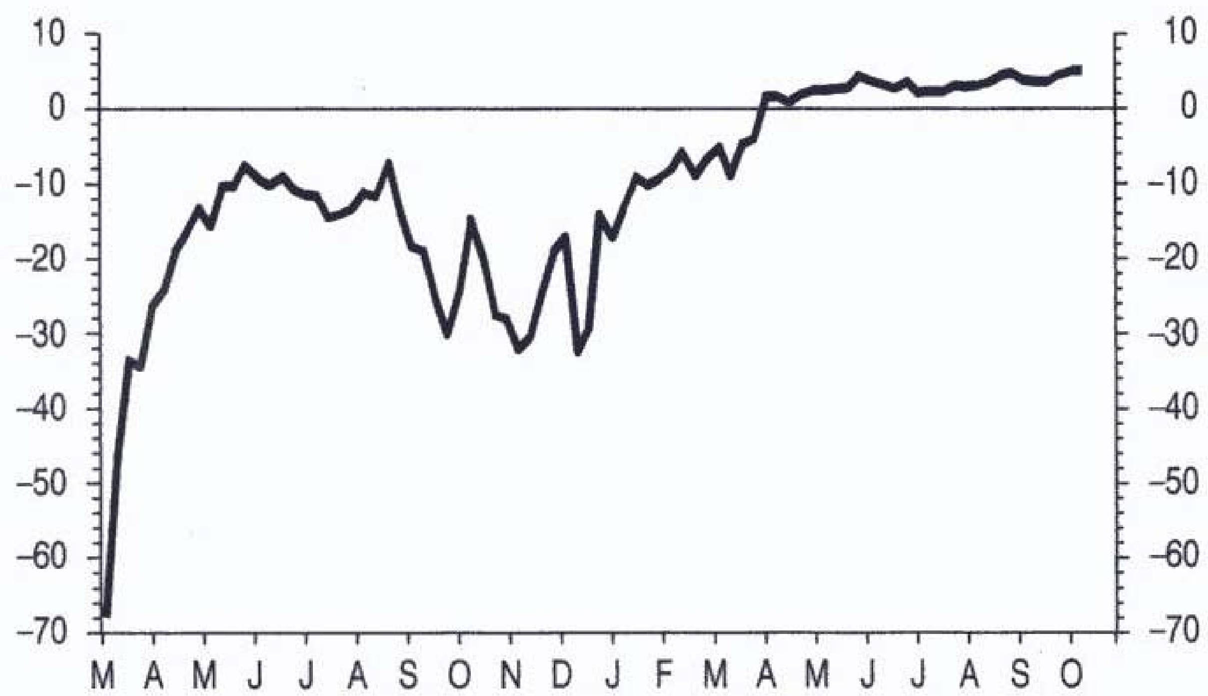


Figure 2.3 3Com/Palm — negative stub value

Data Source: Thomson Datastream.

More examples can be found in the text book on page 163ff.

LTCM (1)

The prominent LTCM case is an excellent example of the risks associated with seemingly arbitrage strategies.
The LTCM managers discovered that

- the share price of Royal Dutch Petroleum at the London exchange
- the share price of Shell Transport and Trading at the New York exchange

do not reflect the parity in earnings and dividends between these two units of the Royal Dutch/Shell holding:

> *The dividends of Royal Dutch are 1.5 times higher than the dividends paid by Shell.*

However, the market prices of these shares did not follow this parity for long time but they followed the local markets' sentiment:

LTCM (3)

This example is most puzzling: buy or sell a portfolio with shares in the proportion 3 : 2 and then to hold this portfolio forever. Doing this one can cash in a gain today while all future obligations in terms of dividends are hedged.

But:

"Markets can behave irrational longer than you can remain solvent."

— Keynes

No-Arbitrage with Short-Sales Constraints

Consider the case of non-negative payoffs and short-sales constraints, $A_s^k \geq 0$ and $\lambda_k^i \geq 0$.
The short-sales restriction may apply to one or more securities. Then, the Fundamental Theorem of Asset Pricing reduces to:

Theorem (FTAP with Short-Sales Constraints)

There is no long-only portfolio $\boldsymbol{\theta} \geq \mathbf{0}$ such that $\boldsymbol{q}'\boldsymbol{\theta} \leq \mathbf{0}$ and $\boldsymbol{A\theta} > \mathbf{0}$ is equivalent to $\boldsymbol{q} \gg \mathbf{0}$.

The proof can be found in the text book on page 167.

No-Arbitrage with Short-Sales Constraints

Hence, *all* positive prices are arbitrage-free: sales restrictions deter rational managers to exploit eventual arbitrage opportunities. Consequently, the no-arbitrage condition does not tell us anything and we need to look at specific assumptions to determine asset prices.

Financial Markets Equilibria

Financial Markets Equilibria (1)

- What we did so far: Derive prices of redundant assets from prices of a set of fundamental assets.
- What we don't know yet: how the prices of the fundamental assets should be related to each other.
- Fundamental Theorem of Asset Prices shows that asset prices are determined by *some* state prices; but the value of the state prices is not determined by the no-arbitrage principle!

Financial Markets Equilibria (2)

Idea: prices are determined by trade – but trades are in turn depending on prices.
The notion of a *competitive equilibrium* captures interdependence of decisions and prices. A competitive equilibrium is a price system such that all agents have optimized their positions and all markets clear.

State prices

As a general rule we obtain that state prices are larger for those states the agents believe to be more likely to occur and in which there are less resources. For special cases like the CAPM, we can get more specific pricing rules.
Asset prices are determined by the expected payoff adjusted by the scarcity of resources. This adjustment is measured by the covariance of the payoffs and the aggregate availability of resources (the market portfolio).

General Risk-Return Tradeoff (1)

Goal: general risk-return formula from the principle of no-arbitrage. CAPM, APT and behavioral CAPM will simply be special cases of this general result.

Recall that the absence of arbitrage is equivalent to the existence of state prices $\boldsymbol{\pi}^*$ such that $R_f = \mathbb{E}_{\pi^*}(R^k)$, for all $k = 1, \ldots, K$.

We define the likelihood ratio process $\ell_s := \pi_s^*/p_s$ to convert this:

$$R_f = \mathbb{E}_{\pi^*}(R^k) = \sum_s \pi_s^* R_s^k = \sum_s p_s \left(\frac{\pi_s^*}{p_s}\right) R_s^k = \sum_s p_s \ell_s R_s^k = \mathbb{E}_p(\ell R^k).$$

General Risk-Return Tradeoff (2)

Recall that by definition of the covariance we can rewrite this to

$$\mathbb{E}_p(R^k) = R_f - \text{cov}_p(R^k, \ell)$$

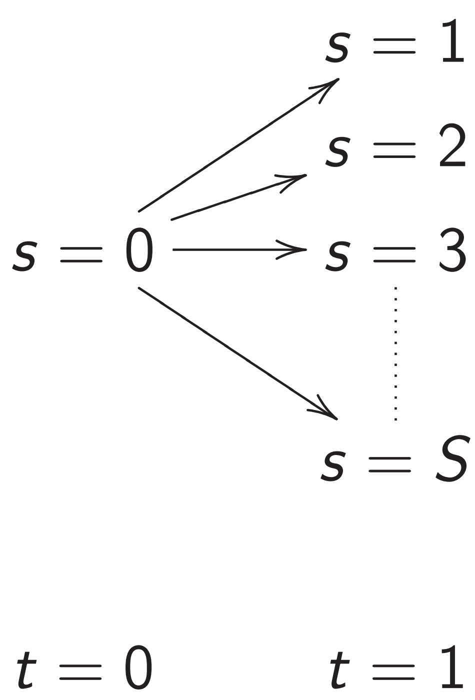
where the covariance of the strategy returns to the likelihood ratio represents the unique risk measure.
Hence, we found a simple risk-return formula which is based on the covariance to a unique factor.

General Risk-Return Tradeoff (3)

Is this the ultimate formula for asset-pricing?
Not really: in a sense we only exchanged one unknown, the state price measure, with another unknown, the likelihood ratio process. The remaining task is to identify the likelihood ratio process based on reasonable economic assumptions.

Definition of Financial Markets Equilibria (1)

The time-uncertainty structure is described by



Definition of Financial Markets Equilibria (2)

- As before, we denote the assets by $k = 0, 1, 2, \ldots, K$. The first asset, $k = 0$, is the risk-free asset delivering the certain payoff 1 in all second period states.
- Each investor $i = 0, hdots, I$ is described by his exogenous wealth in all states of the world $\boldsymbol{w}^i = (w^i_0, \ldots, w^i_S)'$.
- Given these exogenous entities and given the asset prices $\boldsymbol{q} = (q^0, \ldots, q^K)'$ he can finance his consumption $\boldsymbol{c}^i = (c^i_0, \ldots, c^i_S)'$ by trading the assets.
- We denote by $\boldsymbol{\theta}^i = (\theta^{i,0}, \ldots, \theta^{i,K})'$ the vector of asset trade of agent i. Note that $\theta^{i,k}$ can be positive or negative, i.e., agents can buy or sell assets.

Definition of Financial Markets Equilibria (3)

In these terms, the agent's decision problem is:

$$\max_{\boldsymbol{\theta}^i \in \mathbb{R}^{K+1}} U^i(\boldsymbol{c}^i) \quad \text{such that} \quad c_0^i + \sum_{k=0}^{K} q^k \theta^{i,k} = w_0^i$$

$$\text{and} \quad c_s^i = \sum_{k=0}^{K} A_s^k \theta^{i,k} + w_s^i \geq 0, \quad s = 1, \ldots, S.$$

Definition of Financial Markets Equilibria (4)

Considering that some parts of the wealth may be given in terms of assets, this can be written as:

$$\max_{\hat{\theta}^i \in \mathbb{R}^{K+1}} U^i(c^i) \quad \text{such that} \quad c_0^i + \sum_{k=0}^{K} q^k \hat{\theta}^{i,k} = \sum_{k=0}^{K} q^k \theta_A^{i,k} + w_0^i$$

$$\text{and} \quad c_s^i = \sum_{k=0}^{K} A_s^k \hat{\theta}^{i,k} + w_{\perp s}^i, \quad s = 1, \ldots, S.$$

Definition of Financial Markets Equilibria (5)

A *financial markets equilibrium* is a system of asset prices and an allocation of assets such that every agent optimizes his decision problem and markets clear.

Definition of Financial Markets Equilibria (6)

Definition (financial markets equilibrium (FME))

A FME is a list of portfolio strategies $\hat{\boldsymbol{\theta}}^{\mathrm{opt},i}$, $i = 1, \dots, I$, *and a price system* q^k, $k = 0, \dots, K$, *such that for all* $i = 1, \dots, I$,

$$\hat{\boldsymbol{\theta}}^{\mathrm{opt},i} = \arg\max_{\hat{\boldsymbol{\theta}}^i \in \mathbb{R}^{K+1}} U^i(\boldsymbol{c}^i)$$

$$\text{such that} \quad c_0^i + \sum_{k=0}^{K} q^k \hat{\theta}^{i,k} = \sum_{k=0}^{K} q^k \theta_A^{i,k} + w_0^i$$

$$\text{and} \quad c_s^i = \sum_{k=0}^{K} A_s^k \hat{\theta}^{i,k} + w_{\perp s}^i, \quad s = 1, \dots, S,$$

and markets clear:

Definition of Financial Markets Equilibria (6)

Definition (financial markets equilibrium (FME))

A FME is a list of portfolio strategies $\hat{\boldsymbol{\theta}}^{\mathrm{opt},i}$, $i = 1, \dots, I$, and a price system q^k, $k = 0, \dots, K$, such that for all $i = 1, \dots, I$, and markets clear:

$$\sum_{i=1}^{I} \hat{\theta}^{\mathrm{opt},i,k} = \sum_{i=1}^{I} \theta_0^{i,k}, \quad k = 0, \dots, K.$$

Do consumption markets clear?

We only required asset markets to clear. What about markets for consumption?
Can we show that also the sum of the consumption is equal to the sum of the available resources, i.e.,

$$\sum_{i=1}^{I} c_0^i = \sum_{i=1}^{I} w_0^i \quad \text{and} \quad \sum_{i=1}^{I} c_s^i = \sum_{i=1}^{I} \left(\sum_{k=0}^{K} A_s^k \theta_A^{i,k} + w_{\perp s}^i \right),$$

for all $s = 1, \ldots, S$?

Do consumption markets clear?

This follows from the agents' budget restrictions:

$$\sum_{i=1}^{I}\left(c_0^i+\sum_{k=0}^{K}q^k\hat{\theta}^{\mathrm{opt},i,k}\right)=\sum_{i=1}^{I}\left(w_0^i+\sum_{k=0}^{K}q^k\theta_A^{i,k}\right)$$

and

$$\sum_{i=1}^{I}c_s^i=\sum_{i=1}^{I}\left(\sum_{k=0}^{K}A_s^k\hat{\theta}^{\mathrm{opt},i,k}+w_{\perp s}^i\right),\quad s=1,\ldots,S,$$

because asset markets clear: $\sum_{i=1}^{I}\hat{\theta}^{\mathrm{opt},i,k}=\sum_{i=1}^{I}\theta_A^{i,k}$, $k=0,\ldots,K$.
Hence, nothing is missing in the FME definition.

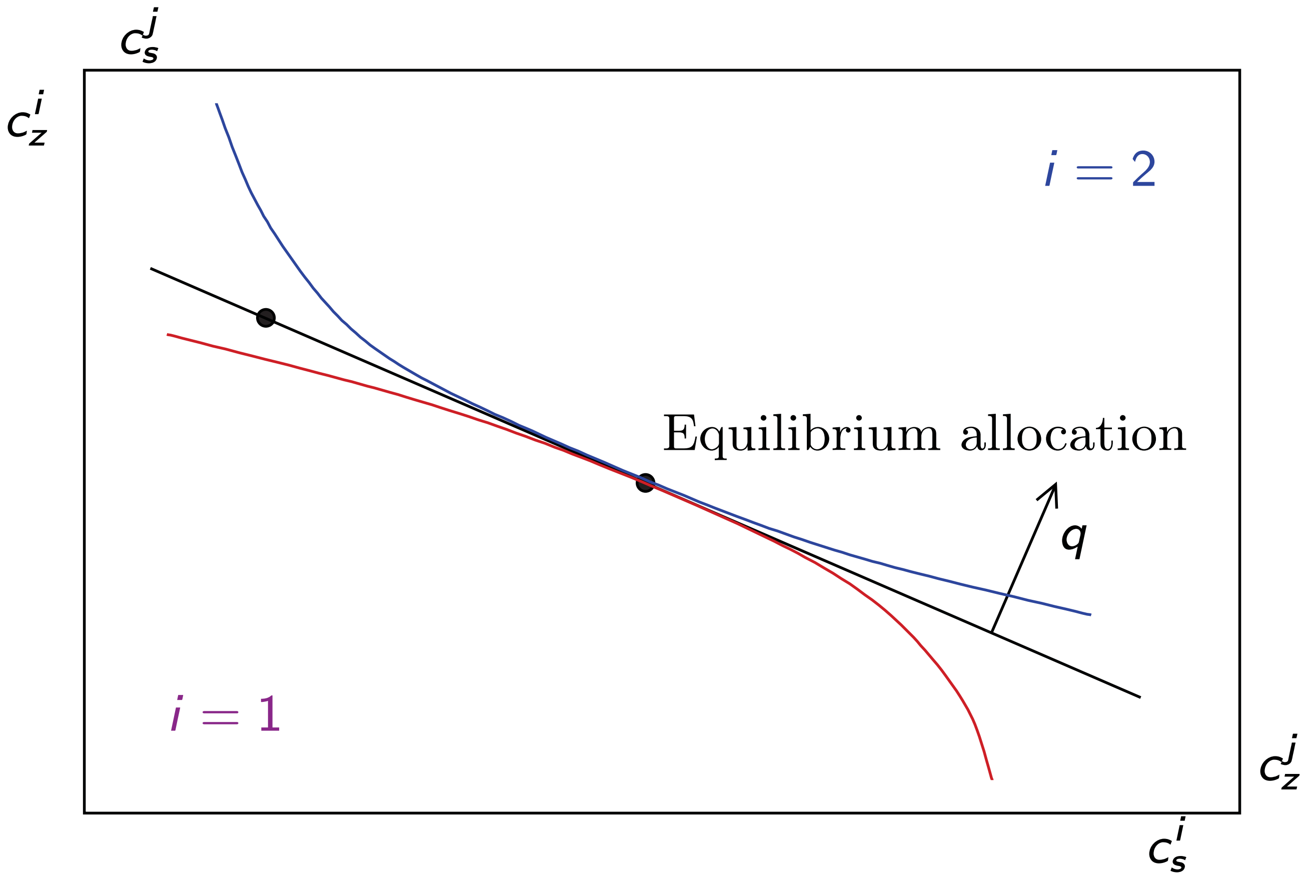
Arbitrage in equilibrium?

- In a financial market equilibrium there cannot be arbitrage opportunities: otherwise the agents would not be able to solve their maximization problem since any portfolio they consider could still be improved by adding the arbitrage portfolio.
- Deriving asset prices from an equilibrium model automatically leads to arbitrage-free prices.

Edgeworth box

A financial markets equilibrium can be illustrated by an Edgeworth Box.
At the equilibrium allocation both agents have optimized their consumption by means of asset trade given their budget constraint and markets clear.

Edgeworth box



Marginal rate of substitution

The Edgeworth Box suggests that asset prices should be related to the agents' marginal rates of substitution.
Investigating the first order conditions for solving their optimization problems, we see that the marginal rates of substitution are one candidate for state prices.

$$q^k = \sum_{s=1}^{S} \underbrace{\frac{\partial c_s U^i(c_0^i, \ldots, c_S^i)}{\partial c_0 U^i(c_0^i, \ldots, c_S^i)}}_{\pi_s^i} A_s^k, \quad k = 0, \ldots, K.$$

Marginal rate of substitution

In particular, for the case of expected utility

$$U^i(c_0^i, \ldots, c_S^i) = u^i(c_0^i) + \delta^i \sum_{s=1}^{S} \mathrm{prob}_s^i u^i(c_s^i)$$

we get:

$$q^k = \sum_{s=1}^{S} \underbrace{\frac{\mathrm{prob}_s^i \delta^i \partial c_s u^i(c_s^i)}{\partial c_0 u^i(c_0^i)}}_{\pi_s^i} A_s^k, \quad k = 0, \ldots, K.$$

Converting into Finance Terms

Financial markets equilibrium in finance terms:

$$\max_{\boldsymbol{\lambda}\in\Delta^{K+2}} U^i(c^i) \quad \text{s. th.} \quad c_0^i = w_0^i - (1-\lambda^c)\sum_{k=0}^{K}\hat{\lambda}^{i,k} w_0^i$$

$$\text{and} \quad c_s^i = \left(\sum_{k=1}^{K} R_s^k \hat{\lambda}^{i,k}\right) w_0^{i,\text{fin}} + w_{\perp s}^i, \quad s = 1, \ldots, S.$$

All together we can define a FME in finance terms – compare Definition 4.8 in the book.

Intertemporal Trade

- Financial market offers intertemporal trade, for savings and loans. Agents have different wealth along their life cycle, which causes demand for savings and loans.
- Interest rates can be explained by demand and supply on the savings and loans market. Interest rates are positive since agents should have a positive time preference: they discount future consumption.
- Finally, one would expect that the aggregate resources relative to aggregate needs also determine interest rates.
- An example for intertemporal trade can be found in the text book on page 175f.

Formal model

Denoting the savings amount by s and the interest rate by r, the decision problem is given by:

$$\max_s u(c_0) + \delta u(c_1) \quad \text{such that} \quad c_0 + s = w_0$$
$$\text{and} \quad c_1 = w_1 + (1+r)s.$$

Eliminating s, the two budget constraints can be combined into a single one written in terms of present values:

$$c_0 + \frac{1}{1+r} c_1 = w_0 + \frac{1}{1+r} w_1.$$

Solution

The first order condition to this problem is:

$$\frac{u'(c_0)}{\delta u'(c_1)} = (1 + r).$$

For the logarithmic utility this leads to a simple theory of interest rates:

$$1 + r = (1 + g)/\delta, \quad \text{where} \quad c_1 = (1 + g)c_0.$$

Hence g is the growth rate of consumption. That is to say, interest rates increase, if people become less patient and if consumption growth increases.
In general interest rates increase when the growth of the GDP is strong and falling interest rates may be a signal for a recession.

Special Cases: CAPM, APT and Behavioral CAPM

Special Cases: CAPM, APT and Behavioral CAPM

The general model can be used to find simple derivations for the CAPM, APT and the Behavioral CAPM. In all of these cases, diversification is the central motive for trading on financial markets.

- Assume that the consumption in the first period is already decided (no time-diversification).
- Assume that all agents agree on the probabilities of occurrence of the states, prob_s, $s = 1, \ldots, S$ (no betting).

Special Cases: CAPM, APT and Behavioral CAPM

Assumptions underlying CAPM

1. There exists a risk-free asset, i.e. $(1, \ldots, 1)' \in \operatorname{span}\{\boldsymbol{A}\}$.
2. There is no first period consumption nor first period endowments.
3. Endowments are spanned, i.e., $(w_1^i, \ldots, w_S^i)' \in \operatorname{span}\{\boldsymbol{A}\}$, $i = 1, \ldots, I$.
4. Expectations are homogeneous, i.e., $\mathrm{prob}_s^i = \mathrm{prob}_s$, $i = 1, \ldots, I$ and $s = 1, \ldots, S$.
5. Preferences are mean-variance, i.e.,

$$U^i(c_1^i, \ldots, c_S^i) = V^i(\mu(c_1^i, \ldots, c_S^i), \sigma(c_1^i, \ldots, c_S^i)),$$

$\mu(c_1^i, \ldots, c_S^i) = \sum_{s=1}^S \mathrm{prob}_s c_s^i,$
$\sigma^2(c_1^i, \ldots, c_S^i) = \sum_{s=1}^S \mathrm{prob}_s (c_S^i - \mu(c_1^i, \ldots, c_S^i))^2.$

Notation

We introduce the following notation:

- $\boldsymbol{A} = (\mathbf{1}, \hat{\boldsymbol{A}})$ where $\hat{\boldsymbol{A}}$ is the $S \times K$ matrix of *risky* assets.
- By $\mu(\hat{\boldsymbol{A}}) = (\mu(\hat{\boldsymbol{A}}^0), \ldots, \mu(\hat{\boldsymbol{A}}^K))$ we denote the vector of mean payoffs of assets in a matrix $\hat{\boldsymbol{A}}$.
- Similarly, $COV(\hat{\boldsymbol{A}}) = (\mathrm{cov}(\boldsymbol{A}^k, \boldsymbol{A}^j))_{k,j=1,\ldots,K}$ denotes (as before) the variance-covariance matrix associated with a matrix $\boldsymbol{A}$.

Note that

$$\sigma^2(\hat{\boldsymbol{A}}\hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}}' \hat{\boldsymbol{A}}' \boldsymbol{\Lambda}(\mathbf{prob}) \hat{\boldsymbol{A}} \hat{\boldsymbol{\theta}} - \mu(\hat{\boldsymbol{A}}\hat{\boldsymbol{\theta}}) \mu(\hat{\boldsymbol{A}}\hat{\boldsymbol{\theta}})' = \hat{\boldsymbol{\theta}}' \,\mathrm{cov}(\hat{\boldsymbol{A}}) \hat{\boldsymbol{\theta}}.$$

Mean-variance decision problem

We analyze the decision problem of a mean-variance agent

$$\max_{\hat{\boldsymbol{\theta}}^i \in \mathbb{R}^{K+1}} V^i(\mu(\boldsymbol{c}^i), \sigma^2(\boldsymbol{c}^i)) \quad \text{such that} \quad \sum_{k=0}^{K} q^k \hat{\theta}^{i,k} = \sum_{k=0}^{K} q^k \theta_A^{i,k} = w^i,$$

where $c_s^i := \sum_{k=0}^{K} A_s^k \hat{\theta}^{i,k}$, $s = 1, \ldots, S$.

- Recall that $q^0 := 1/R_f$.
- From the budget equation we can then express the units of the risk free asset held by $\hat{\theta}^0 = R_f(w^i - \hat{\boldsymbol{q}}'\hat{\boldsymbol{\theta}})$.

Hence, we can re-write the maximization problem as

$$\max_{\hat{\boldsymbol{\theta}}^i \in \mathbb{R}^K} V^i \left(R_f w^i + (\mu(\hat{\boldsymbol{A}}) - R_f \hat{\boldsymbol{q}})' \hat{\boldsymbol{\theta}}^i, \sigma^2(\hat{\boldsymbol{A}} \hat{\boldsymbol{\theta}}^i) \right).$$

Solution (1)

The first order condition is:

$$\mu(\hat{\boldsymbol{A}}) - R_f \hat{\boldsymbol{q}} = \rho^i COV(\hat{\boldsymbol{A}}) \hat{\boldsymbol{\theta}}^i,$$

where $\rho^i := \frac{\partial_\sigma V^i}{\partial_\mu V^i}(\mu, \sigma^2)$ is the agent's degree of risk aversion. Solving for the portfolio we obtain

$$\hat{\theta}^i = \frac{1}{\rho^i} COV(\hat{\boldsymbol{A}})^{-1} (\mu(\hat{\boldsymbol{A}}) - R_f \hat{\boldsymbol{q}}).$$

because the first order condition is a linear system of equations differing across agents only by a scalar, ρ^i.

Solution (1)

This is again the *two-fund separation property*. We see that any two different agents, i and i', will form portfolios whose ratio of risky assets,

$$\hat{\theta}^{i,k}/\hat{\theta}^{i,k'} = \hat{\theta}^{i',k}/\hat{\theta}^{i',k'},$$

are identical.

Solution (2)

Dividing the first order condition by ρ^i and summing up over all agents we obtain:

$$\left(\sum_i \frac{1}{\rho^i}\right)\left(\mu(\hat{\mathbf{A}}) - R_f \hat{\mathbf{q}}\right) = \operatorname{cov}(\hat{\mathbf{A}}) \sum_i \hat{\boldsymbol{\theta}}^i.$$

- We know that
$$\sum_i \hat{\boldsymbol{\theta}}^i = \sum_i \boldsymbol{\theta}^i_A =: \hat{\boldsymbol{\theta}}^M,$$
denoted by asset M, the market portfolio.
- Denote the market portfolio's payoff by $\hat{\mathbf{A}}^M = \hat{\mathbf{A}}\hat{\boldsymbol{\theta}}^M$.
- Let the price of the market portfolio be $\hat{q}^M = \hat{\mathbf{q}}'\hat{\boldsymbol{\theta}}^M$.

Then we get:

$$\left(\mu(\hat{\mathbf{A}}) - R_f \hat{\mathbf{q}}\right) = \left(\sum_i \frac{1}{\rho^i}\right)^{-1} \operatorname{cov}(\hat{\mathbf{A}})\hat{\boldsymbol{\theta}}^M.$$

Solution (3)

$$\left(\mu(\hat{\boldsymbol{A}}) - R_f \hat{\boldsymbol{q}}\right) = \left(\sum_i \frac{1}{\rho^i}\right)^{-1} \operatorname{cov}(\hat{\boldsymbol{A}})\hat{\boldsymbol{\theta}}^M.$$

Multiplying both sides with the market portfolio yields

$$\left(\sum_i \frac{1}{\rho^i}\right)^{-1} = \frac{\left(\mu(\hat{\boldsymbol{A}}^M) - R_f \hat{q}^M\right)}{\sigma^2(\hat{\boldsymbol{A}}^M)}.$$

Substituting this back into the former equation we finally get the asset pricing rule:

$$R_f \hat{\boldsymbol{q}} = \mu(\hat{\boldsymbol{A}}) - \frac{\left(\mu(\hat{\boldsymbol{A}}^M) - R_f \hat{q}^M\right)}{\sigma^2(\hat{\boldsymbol{A}}^M)} \operatorname{cov}(\hat{\boldsymbol{A}}, \hat{\boldsymbol{A}}^M).$$

Solution (4)

$$R_f \hat{\boldsymbol{q}} = \mu(\hat{\boldsymbol{A}}) - \frac{\left(\mu(\hat{\boldsymbol{A}}^M) - R_f \hat{q}^M\right)}{\sigma^2(\hat{\boldsymbol{A}}^M)} \operatorname{cov}(\hat{\boldsymbol{A}}, \hat{\boldsymbol{A}}^M).$$

Writing this more explicitly we have derived:

$$q^k = \frac{\mu(\boldsymbol{A}^k)}{R_f} - \frac{\operatorname{cov}(\boldsymbol{A}^k, \boldsymbol{A}^M)}{\operatorname{var}(\boldsymbol{A}^M)} \left(\frac{\mu(\boldsymbol{A}^M)}{R_f} - q^M \right).$$

We see that the preset price of an asset is given by its expected payoff discounted to the present minus a risk premium that increases the higher the covariance to the market portfolio.

Solution (5)

To derive the analog in finance terms, multiply the resulting expression by R_f and divide it by q^k and q^M. We obtain

$$\mu(R^k) - R_f = \beta^k(\mu(R^M) - R_f) \quad \text{where} \quad \beta^k = \frac{\operatorname{cov}(R^k, R^M)}{\sigma^2(R^M)},$$

This is the classical CAPM formula, compare Sec. 3.2.1.

An alternative derivation of the CAPM using the likelihood Ratio Process can be found in the text book on page 180ff.

Arbitrage Pricing Theory (APT)

The APT is a generalization of the CAPM in which the likelihood ratio process is a linear combination of many factors.

- Let $R^1, \ldots, R^F$ be the returns that the market rewards for holding the F factors $f = 1, \ldots, F$, i.e., let $\ell \in \text{span}\{\mathbf{1}, \boldsymbol{R}^1, \ldots, \boldsymbol{R}^F\}$.
- Following the same steps as before we get

$$\mathbb{E}_p(R^k) - R_f = \sum_{f=1}^{F} b^f \left(\mathbb{E}_p(R^f) - R_f\right).$$

This gives more flexibility for an econometric regression. But can we give an economic foundation to it?

Deriving the APT in the CAPM with Background Risk (1)

The main idea is that the APT can be thought of as a CAPM with background risk.
We need to prove that $\boldsymbol{\ell} \in \text{span}\left\{\mathbf{1}, \boldsymbol{R}^1, \ldots, \boldsymbol{R}^F\right\}$ with $\text{cov}_p(R^f, R^{f'}) = 0$ for $f \neq f'$. Note that one of the factors may be the market itself, i.e., $f = M$ so that the APT is a true generalization of the CAPM.

Deriving the APT in the CAPM with Background Risk (2)

We consider again the maximization problem

$$\max_{\hat{\boldsymbol{\theta}}^i \in \mathbb{R}^{K+1}} V^i\left(c_0^i, \mu(\boldsymbol{c}_1^i), \sigma^2(\boldsymbol{c}_1^i)\right)$$

$$\text{such that} \quad c_0^i + \sum_{k=0}^{K} q^k \hat{\theta}^{i,k} = w_0^i + \sum_{k=0}^{K} q^k \theta_A^{i,k},$$

where $\boldsymbol{c}_1^i = \boldsymbol{w'}_{\perp 1}^i + \sum_{k=0}^K \boldsymbol{A}^k \hat{\theta}^{i,k}$.

Deriving the APT in the CAPM with Background Risk (3)

In terms of state prices the budget restriction can be written as:

$$c_0^i + \sum_{s=1}^{S} \pi_s^* c_s^i = w_0^i + \sum_{s=1}^{S} \pi_s^* w_s^i$$

and as before $\mathbf{c}_1^i - \mathbf{w'}_{\perp 1}^i \in \text{span}\,\{\mathbf{A}\}$.
Using the likelihood ratio process, the budget restriction becomes:

$$c_0^i + \sum_{s=1}^{S} p_s \ell_s c_s^i = w_0^i + \sum_{s=1}^{S} p_s \ell_s w_s^i \quad \text{and} \quad (\mathbf{c}_1^i - \mathbf{w'}_{\perp 1}^i) \in \text{span}\,\{\mathbf{A}\}\,,$$

Deriving the APT in the CAPM with Background Risk (4)

Next we will show that $(\boldsymbol{c}_1^i - \boldsymbol{w}'^i_{\perp 1}) \in \text{span}\,\{\mathbf{1}, \boldsymbol{\ell}\}$.

- Suppose
$$(\boldsymbol{c}_1^i - \boldsymbol{w}'^i_{\perp 1}) = a^i \mathbf{1} + b^i \boldsymbol{\ell} + \boldsymbol{\xi}^i,$$
where $\boldsymbol{\xi}^i \notin \text{span}\,\{\mathbf{1}, \boldsymbol{\ell}\}$, i.e., $\mathbb{E}_p(1\xi^i) = \mathbb{E}_p(\ell\xi^i) = 0$.
- Since $\boldsymbol{c}_1^i$ is an optimal portfolio it satisfies the budget and the spanning constraint.

Deriving the APT in the CAPM with Background Risk (5)

Now what would happen if we canceled $\boldsymbol{\xi}^i$ from the agent's demand?

- Since $\mathbb{E}_p(\ell\xi^i) = 0$, also $a^i\mathbf{1} + b^i\boldsymbol{\ell}$ satisfies the budget constraint and obviously $(a^i\mathbf{1} + b^i\boldsymbol{\ell}) \in \text{span}\,\{\boldsymbol{A}\}$.
- $\boldsymbol{\xi}^i$ does not increase the mean consumption, because $\mathbb{E}_p(1\xi^i) = 0$.
- However, $\boldsymbol{\xi}^i$ increases the variance of the consumption, since

$$\begin{aligned}\mathrm{var}_p(c^i) &= \mathrm{var}_p(a^i 1 + b^i \ell + \xi^i) \\ &= (b^i)^2 \,\mathrm{var}_p(\ell) + \mathrm{var}_p(\xi^i) + 2b^i \,\mathrm{cov}_p(\ell, \xi^i)\end{aligned}$$

and

$$\mathrm{cov}_p(\ell, \xi^i) = \mathbb{E}_p(\ell\xi^i) - \mathbb{E}_p(\ell)\mathbb{E}_p(1\xi^i) = 0.$$

Hence it is best to choose $\boldsymbol{\xi}^i = \mathbf{0}$.

Deriving the APT in the CAPM with Background Risk (6)

It remains to argue that the factors can explain the likelihood ratio process:
Aggregating

$$(\boldsymbol{c}_1^i - \boldsymbol{w'}_{\perp 1}^i) = a^i \mathbf{1} + b^i \boldsymbol{\ell}$$

over all agents gives

$$\boldsymbol{\ell} \in \text{span}\{\mathbf{1}, \boldsymbol{R}^M, \tilde{\boldsymbol{R}}^1, \ldots, \tilde{\boldsymbol{R}}^F\},$$

where $\tilde{\boldsymbol{R}}^1, \ldots, \tilde{\boldsymbol{R}}^F$ are F factors that span the non-market risk embodied in the aggregate wealth:

$$\sum_{i=1}^{I} \boldsymbol{w'}_{\perp 1}^i = \sum_{f=1}^{F} \beta^f \tilde{\boldsymbol{A}}^f.$$

Behavioral CAPM (1)

Finally, we want to show how Prospect Theory can be included into the CAPM to build a Behavioral CAPM, a B-CAPM.

- In contrast to the B-CAPM of Chap. 3, we now include behavioral aspects into the consumption based CAPM.
- To do so we assume that the investor has the quadratic Prospect Theory utility

$$v(c_s - RP) := \begin{cases} (c_s - RP) - \frac{\alpha^+}{2}(c_s - RP)^2 & \text{if } c_s > RP, \\ \lambda\left((c_s - RP) - \frac{\alpha^-}{2}(c_s - RP)^2\right) & \text{if } c_s < RP, \end{cases}$$

  and no probability weighting.
- A piecewise quadratic utility is convenient because it contains the CAPM as a special case when $\alpha^+ = \alpha^-$ and $\lambda = 1$.

Behavioral CAPM (2)

Start from the general risk-return decomposition

$$\mathbb{E}(R^k) = R_f - \mathrm{cov}(R^k, \ell).$$

The likelihood ratio process for the piecewise quadratic utility is:

$$\delta^i u'(c_0)\ell(c_s) = \begin{cases} 1 - \alpha^+ c_s & \text{if } c_s > RP, \\ \lambda(1 - \alpha^- c_s) & \text{if } c_s < RP. \end{cases}$$

Now suppose that $c_s = R^M$ holds and that the reference point is the risk-free rate R_f.

Behavioral CAPM (3)

We abbreviate $\hat{\alpha}^{\pm} := \alpha^{\pm}/(\delta^i u'(c_0))$ and denote

$$\mathcal{P}(R^M > R_f) := \sum_{R_s^M > R_f} p_s,$$

$$\text{cov}^+(R^k, R^M) := \sum_{R_s^M > R_f} \frac{p_s}{\mathcal{P}(R^M - F_f)}(R_s^k - \mathbb{E}(R^k))(R_s^M - \mathbb{E}(R^M)),$$

$$\text{cov}^-(R^k, R^M) := \sum_{R_s^M < R_f} \frac{p_s}{\mathcal{P}(R^M - F_f)}(R_s^k - \mathbb{E}(R^k))(R_s^M - \mathbb{E}(R^M)).$$

Behavioral CAPM (4)

Then the general risk-return decomposition is

$$\mathcal{P}(R^M > R_f)\big(\mathbb{E}^+(R^k) - R_f + \hat{\alpha}^+ \operatorname{cov}^+(R^k, R^M)\big)$$
$$+ (1 - \mathcal{P}(R^M > R_f))\lambda\big(\mathbb{E}^-(R^k) - R_f + \hat{\alpha}^- \operatorname{cov}^-(R^k, R^M)\big) = 0.$$

Here $\mathbb{E}^+$ and $\mathbb{E}^-$ denote conditional expectation above and below the risk-free rate.
We see that if $\alpha^+ = \alpha^-$ and $\beta = 1$, we get the CAPM.

Pareto efficiency

Efficiency

Informational efficiency:
Efficient Market Hypothesis, EMH: *In any point in time prices already reflect all public information.* — Eugene Fama

(Compare Chap. 7.)

Allocation efficiency:
Pareto-efficiency: "Nobody can do better without somebody being worse off." — Vilfredo Pareto

Why is Pareto efficiency interesting in finance? Please check in the text book on page 185f!

Derivation (1)

Rewrite the decision problem in terms of state prices instead of asset prices.

$$\max_{\boldsymbol{\theta}\in\mathbb{R}^{K+1}} U(c_0,\ldots,c_s) \quad \text{s.t.} \quad c_0 + \sum_{k=0}^{K} q^k\theta^k = w_0$$

$$\text{and} \quad c_s = \sum_{k=0}^{K} A_s^k\theta^k + w_s \geq 0, \quad s = 1,\ldots,S.$$

Derivation (2)

Substituting the asset prices from the no-arbitrage condition

$$\pi_0 q^k = \sum_{s=1}^{S} \pi_s A_s^k, \quad k = 0, \ldots, K,$$

the budget restrictions can be rewritten as:

$$\pi_0 c_0 + \sum_{s=1}^{S} \pi_s c_s = \pi_0 w_0 + \sum_{s=1}^{S} \pi_s w_s,$$

and

$$c_s - w_s = \sum_{k=0}^{K} A_s^k \theta^k, \quad s = 1, \ldots, S, \quad \text{for some } \theta.$$

Derivation (3)

Is there a better feasible allocation?
An allocation is feasible if it is compatible with the consumption sets of the agents and it does not use more resources than there are available in the economy.

Theorem (First Welfare Theorem)

In a complete financial market the allocation of consumption streams, $(\mathbf{c}^i)_{i=1}^I$, is Pareto-efficient, i.e., there does not exist an alternative attainable allocation of consumption $(\hat{\mathbf{c}}^i)_{i=1}^I$, such that no consumer is worse off and some consumer is better off, i.e., $U^i(\hat{\mathbf{c}}^i) \geq U^i(\mathbf{c}^i)$ for all i and $U^i(\hat{\mathbf{c}}^i) > U^i(\mathbf{c}^i)$ for some i.

The proof can be found in the text book on page 187.

Pareto efficiency and completeness

- In general: Pareto efficiency does not hold in incomplete markets.
- In special situations, however, it holds, e.g. if the utility functions of the agents are "similar" to each other (see the book for details!).

Aggregation

Aggregation

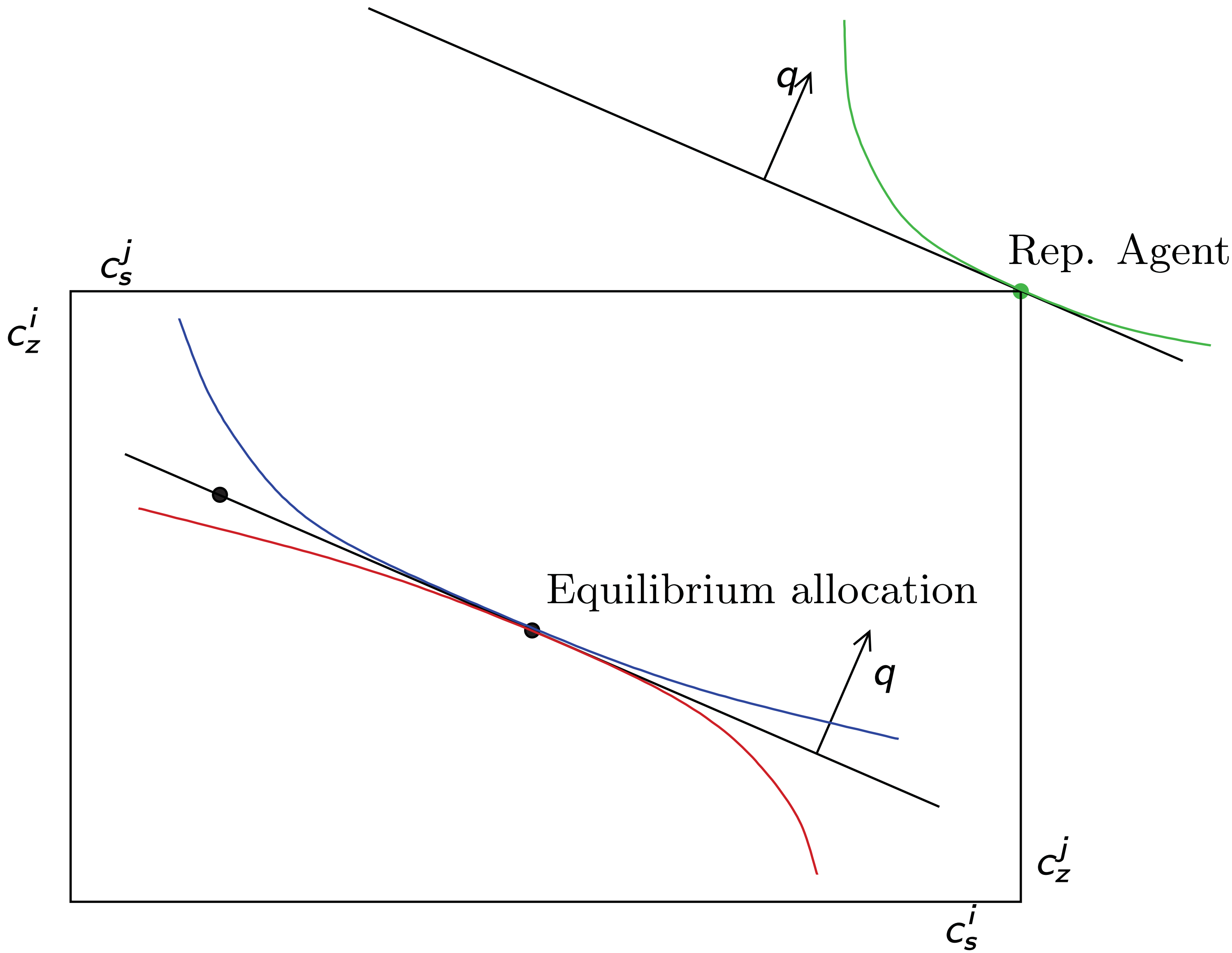
In this section we answer the following questions of increasing difficulty:

1. Under which conditions can prices which are market aggregates be generated by aggregate endowments (consumption) and some aggregate utility function?
2. Is it possible to find an aggregate utility function that has the same properties as the individual utility functions?
3. Is it possible to use the aggregate decision problem to determine asset prices "out of sample"?

Representative Agent in the Edgeworth Box

At the equilibrium allocation asset prices are determined by the trade of two agents, however, they could also be thought of as being derived from a single utility function that is maximized over the budget set based on aggregate endowments.

Representative Agent in the Edgeworth Box



"Anything Goes"

The answer to our first question is even simpler since it does not need any information on the individual's utility functions:

Theorem (Anything Goes Theorem)

Let $\mathbf{q}$ be an arbitrage-free asset price vector for the market structure $\mathbf{A}$. Then there exists an economy with a representative consumer maximizing an expected utility function such that $\mathbf{q}$ is the equilibrium price vector of this economy.

The proof can be found in the text book on page 189.

Aggregate and Individual Utility Functions

Let us now discuss our question 2:

> *Is it possible to find an aggregate utility function that has the same properties as the individual utility functions?*

Derivation (1)

- Note first that Pareto-efficiency is equivalent to maximizing some *welfare function*. It assigns a social utility to each allocation.
- Let $\gamma^i > 0$ be the weight of agent i in the social welfare function $\sum_{i=1}^{I} \gamma^i U^i(c^i)$.
- Choosing the welfare weights $\gamma^i > 0$ equal to the reciprocal of the agents' marginal utility of consumption in period 0 attained in the financial market equilibrium,

$$\gamma^i = \frac{1}{\partial_0 U^i(\overset{*}{c}{}^i)},$$

one can generate the equilibrium consumption allocation from the social welfare function.

Derivation (2)

Recall that under differentiability and boundary assumptions Pareto-efficiency implies

$$\frac{\nabla_1 U^1(\overset{*}{c}^1)}{\partial_0 U^1(\overset{*}{c}^1)} = \ldots = \frac{\nabla_1 U^I(\overset{*}{c}^I)}{\partial_0 U^I(\overset{*}{c}^I)} =: \pi.$$

Derivation (3)

Define

$$U^R(\boldsymbol{W}) := \sup_{\boldsymbol{c}^1,\ldots,\boldsymbol{c}^I} \left\{ \sum_{i=1}^{I} \gamma^i U^i(\boldsymbol{c}^i) \,\middle|\, \sum_{i=1}^{I} \boldsymbol{c}^i = \boldsymbol{W} \right\}$$

where $\gamma^i = 1/(\partial_0 U^i(\overset{*}{\boldsymbol{c}}{}^i))$.
The FOC for this maximization problem is

$$\gamma^1 \nabla U^1(\overset{*}{\boldsymbol{c}}{}^1) = \ldots = \gamma^I \nabla U^I(\overset{*}{\boldsymbol{c}}{}^I) =: \boldsymbol{\lambda} \quad \text{and} \quad \nabla U^R(\boldsymbol{W}) = \boldsymbol{\lambda}.$$

Hence:

$$\nabla_1 U^R(\boldsymbol{W}) = \frac{\nabla_1 U^i(\overset{*}{\boldsymbol{c}}{}^i)}{\partial_0 U^i(\overset{*}{\boldsymbol{c}}{}^i)} \quad \text{and} \quad \partial_0 U^R(\boldsymbol{W}) = 1.$$

Derivation (4)

Consider

$$\max_{\boldsymbol{\theta}} U^R(\boldsymbol{c}^R) \quad \text{such that } \boldsymbol{c}^R - \boldsymbol{W} \leq \begin{pmatrix} -\boldsymbol{q}^* \\ \boldsymbol{A} \end{pmatrix} \boldsymbol{\theta}.$$

The FOC is

$$\boldsymbol{q}^{*\prime} = \frac{\nabla_1 U^R(\overset{*}{\boldsymbol{c}}{}^R)'}{\partial_0 U^R(\overset{*}{\boldsymbol{c}}{}^R)} \boldsymbol{A} = \frac{\nabla_1 U^i(\overset{*}{\boldsymbol{c}}{}^i)'}{\partial_0 U^i(\overset{*}{\boldsymbol{c}}{}^i)} \boldsymbol{A} = \pi' \boldsymbol{A}.$$

Note that $\boldsymbol{c}^R = \boldsymbol{W}$, the aggregate wealth of the economy.

What Does Aggregate Utility Tell Us?

Postulating some utility function of the representative agent we can now test whether asset prices are in line with optimization by referring to aggregate consumption data. But when does the aggregate utility function really represent the individuals?
We first prove: if all individual utility functions are of the expected utility type with common time preference and common beliefs, then the representative agent is also an expected utility maximizer with the same time preference and the same beliefs.

Aggregation of Similar Utilities (1)

Proposition

Assume that for all $i = 1, \dots, I$ the utility functions u^i agree and that the time discounting δ is also independent of i. Moreover assume that the beliefs p_s, $s = 1, \dots, S$, are homogeneous, i.e., let U^i be given by

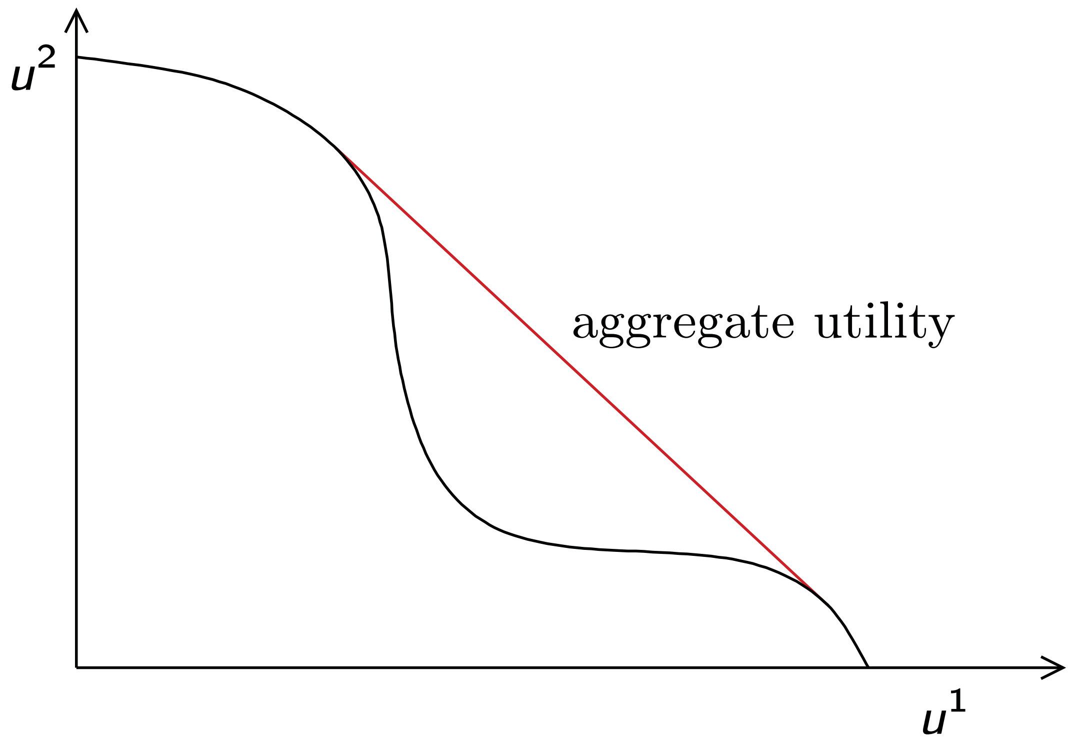
$$U^i(\mathbf{c}^i) = u^i(c_0^i) + \beta \sum_{s=1}^{S} p_s u^i(c_s^i) \quad \text{for } i = 1, \dots, I.$$

Then $U^R(\mathbf{c}^R) = u^R(\mathbf{c}^R) + \beta \sum_{s=1}^{S} p_s u^R(c_s^R)$, for some function $u^R : \mathbb{R} \to \mathbb{R}$.

The proof can be found in the text book on page 191.

The Prospect Theory Case

In the case of Prospect Theory the representative agent may not need to be risk loving over losses since this non-concavity of the utility gets smoothed out by the maximization.



Limitations

Taking the representative agent perspective one can even forget about non-concavities in the individual utility functions.
But unfortunately the representative agent technique has a natural limitation: it is generally not useful to tell us anything about asset prices that we do not know yet (out of sample predictions).

When Does Aggregation Tell Us Something New?

Is it possible to use the aggregate decision problem to determine asset prices 'out of sample', i.e., after some change of e.g., the dividend payoffs? ("demand aggregation")
The answer is yes, but only if we have either:

1. Identical utility functions and identical endowments
2. Quasi-linearity: $U^i(c_0^i, \ldots, c_S^i) = c_0^i + u^i(c_1^i, \ldots, c_s^i)$
3. Expected utility with common beliefs *and*
    1. no-aggregate risk $\sum_{k=1}^K A_s^k = \sum_{k=1}^K A_z^k$ for all s, z *or*
    2. complete markets *and*
        - CRRA and collinear endowments *or*
        - identical CRRA *or*
        - quadratic utility functions

[Hens and Pilgrim, 2003]
Examples on how the heterogenous preferences get aggregated can be found in the text book on page 193f.

Empirical Properties of the Representative Agent

We now *assume* that market prices are generated by an individual decision problem.
Which utility function is compatible with the empirical findings in asset prices?

The utility of the representative investor must have CRRA in the long run [Campbell and Viceira, 2002, page 24].

Determining the Risk-Aversion (1)

Let's write the utility function as $u(w) := w^{1-\alpha}/(1-\alpha)$.
An upper bound for α can be found from the first-order condition of utility maximization since, the Sharpe ratio is bounded above by the volatility of the agent's consumption growth.
The upper bound is called the Hansen and Jagannathan bound.
Let $\zeta := \ell/R_f$. Then the no-arbitrage condition reads $\mathbb{E}(\zeta R^k) = 1$. we can write:

$$1 = \mathbb{E}(\zeta R^k) = \mathbb{E}(\zeta)\mathbb{E}(R^k) + \mathrm{corr}(\zeta, R^k)\sigma(\zeta)\sigma(R^k),$$

hence

$$\mathbb{E}(R^k) = R_f - \mathrm{corr}(\zeta, R^k)\frac{\sigma(\zeta)}{\mathbb{E}(\zeta)}\sigma(R^k).$$

Determining the Risk-Aversion (2)

Since the correlation is bounded between -1 and $+1$, we get the inequality:

$$\frac{\left|\mathbb{E}(R^k) - R_f\right|}{\sigma(R^k)} \leq \frac{\sigma(\zeta)}{\mathbb{E}(\zeta)}.$$

In the consumption based asset pricing model with expected utility, we have

$$q^k = u'(c_0)^{-1} \frac{1}{1+\delta} \mathbb{E}\left(u'(c_1) A^k\right),$$

hence

$$\zeta = \frac{1}{1+\delta} \frac{u'(c_1)}{u'(c_0)}.$$

Determining the Risk-Aversion (3)

And in the case of CRRA we get:

$$\zeta = \frac{1}{1+\delta}\left(\frac{c_1}{c_0}\right)^{-\alpha},$$

so that

$$\frac{\left|\mathbb{E}(R^k) - R_f\right|}{\sigma(R^k)} \leq \sigma\left(\frac{1}{1+\delta}\left(\frac{c_1}{c_0}\right)^{-\alpha}\right).$$

Determining the Risk-Aversion (4)

Hens and Wöhrmann estimate the Sharpe ratio of the S&P 500 sampled from annual data since 1973 and compare it with the Hansen-Jagannathan bounds:

Risk aversion	Consumption SDF
1	0.0165
2.5	0.0415
5	0.0844
8	0.1376
10	0.1743
18	0.3311
20	0.3730
30	0.5987

The actual Sharpe ratio in that data is about 0.328. Hence, a relative risk aversion of about 18 would explain the risk adjusted equity premium.

Equity Premium Puzzle

This value is much higher than estimates of individuals' risk aversion using questionnaire techniques (Sec. 2.2.4).
This is called the equity premium puzzle. However, such elicitations crucially depend on the assumptions on a person's wealth level.

Equity Premium Puzzle

A typical question:

> *Consider a fair lottery where you have a 50% chance of doubling your income, and a 50% chance of losing x% of your income. What is the highest loss x that you would be willing to incur to agree to taking part in this lottery?*

Typical answer to this question: $x \approx 23\%$. Based on a CRRA utility function we get an α of 3.22:

$$0.5\frac{(2W)^{1-\alpha}}{1-\alpha} + 0.5\frac{\big((1-x)W\big)^{1-\alpha}}{1-\alpha} = \frac{W^{1-\alpha}}{1-\alpha}.$$

This gives $2^{1-\alpha} + (1-x)^{1-\alpha} = 2$ or $1-\alpha = -2.22$.

A Possible Explanation of the Puzzle

In the above derivation of α we have implicitly assumed that the money at stake is the whole wealth of a person. However, in the question "only" the whole *salary* is at stake.
Assume that the person's background wealth is 50% of his/her salary, then the degree of risk aversion is

$$0.5\frac{\big((0.5+2)w\big)^{1-\alpha}}{1-\alpha}+0.5\frac{\big((0.5+(1-x))w\big)^{1-\alpha}}{1-\alpha}=\frac{\big((0.5+1)w\big)^{1-\alpha}}{1-\alpha}.$$

If we set $x := 23\%$, we obtain $\alpha = 21$, and the alpha increases even more with higher background wealth.

Short-Run Utility (1)

- We have seen: in the long run CRRA.
- But: "In the long run we are all dead." — Keynes

Therefore we study short-run effects and the utility on which they are based.

Short-Run Utility (2)

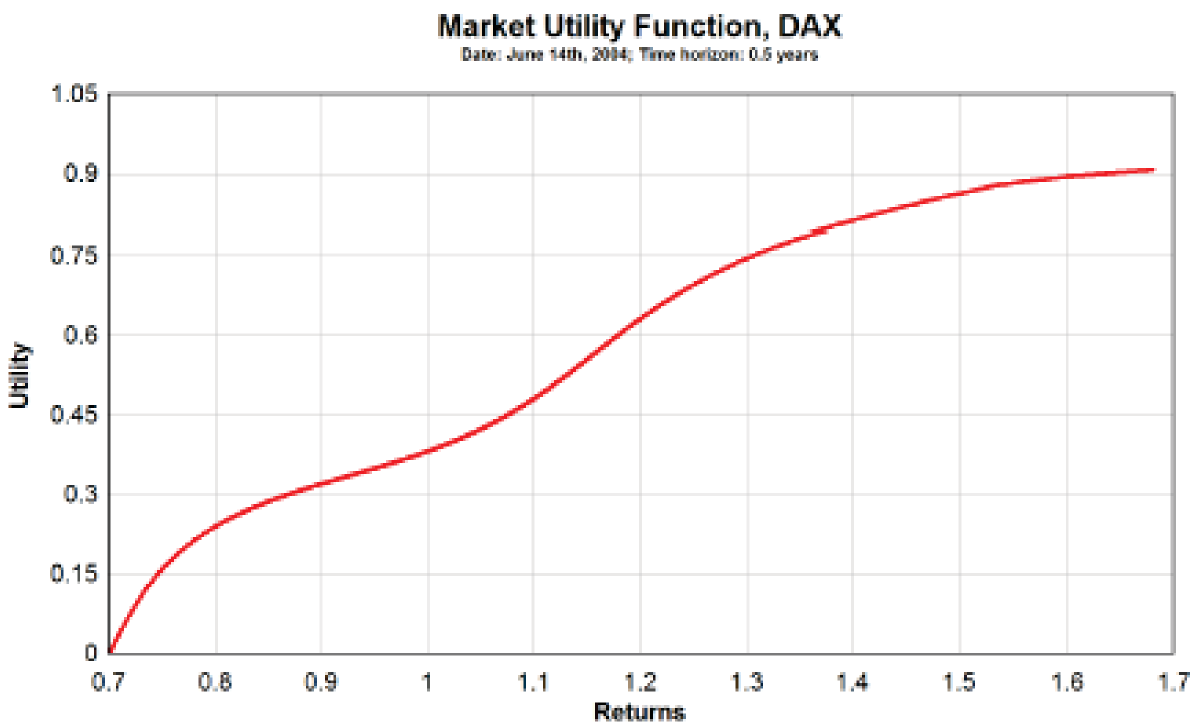
Again we use the first-order condition for optimal investment decisions as a starting point:

$$\ell_s = R^f \frac{u'(c_s)}{(1+\delta)u'(c_0)}, \quad s = 1, \ldots, S.$$

We estimate the likelihood ratio process from observed stock market returns, which fix the p_s's and from option price data which determines the π_s's.

Short-Run Utility (3)

We take the risk free rate, the discount factor and the consumption growth as before. Following this approach [Moro et al.,] estimate the utility function on DAX-data:



Short *and* long run

- In the long run the representative agent is a CRRA maximizer.
- In the short run he is described by Prospect Theory.

A synthesis of the long run and the short run view can be found in the text book on page 199ff.

Dynamics and Stability of Equilibria

Types of Adjustment

We consider three types of dynamics:

- Short-run dynamics: intraday adjustment of market prices.
- Medium-term dynamics: market behavior "off" the equilibrium.
- Long-term dynamics: what investment strategies are successful enough to "stay in the market"?

In this chapter we discuss first the short-term dynamics.

Why is Stability not Obvious?

Stability is not given: whereas on a single market, Adam Smith's "invisible hand" is driving prices towards equilibrium, this is not clear when different markets are intertwined, because the demand of any asset also depends on the price of other assets. (Natural consequence of diversifying agents.)
Nevertheless, we can prove stability in some cases.

A Stable Case

With simple mean-variance preferences of the form

$$U^i(c_1^i, \ldots, c_S^i) := \mu^i(c_1^i, \ldots, c_S^i) - \frac{\gamma^i}{2}\sigma^{2,i}(c_1^i, \ldots, c_S^i),$$

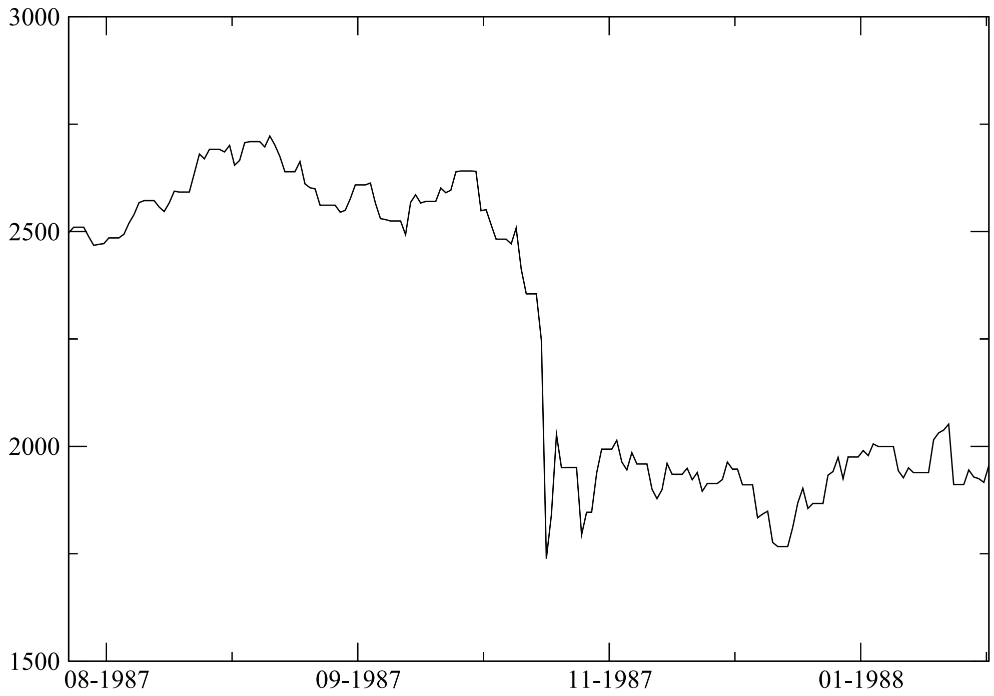
we can prove global stability of the unique financial market equilibrium. This case is obtained, e.g., if utility functions are of the CARA-type and returns are log-normally distributed.

An Instable Case

If however mean variance preferences are obtained from normally distributed returns and Prospect Theory preferences then the mean-variance utility looks more complicated than that, see [De Giorgi and Post, 2005].
Thus, CAPM equilibria may be unstable: due to exogenous shocks they may jump from one possible equilibrium to another equilibrium.

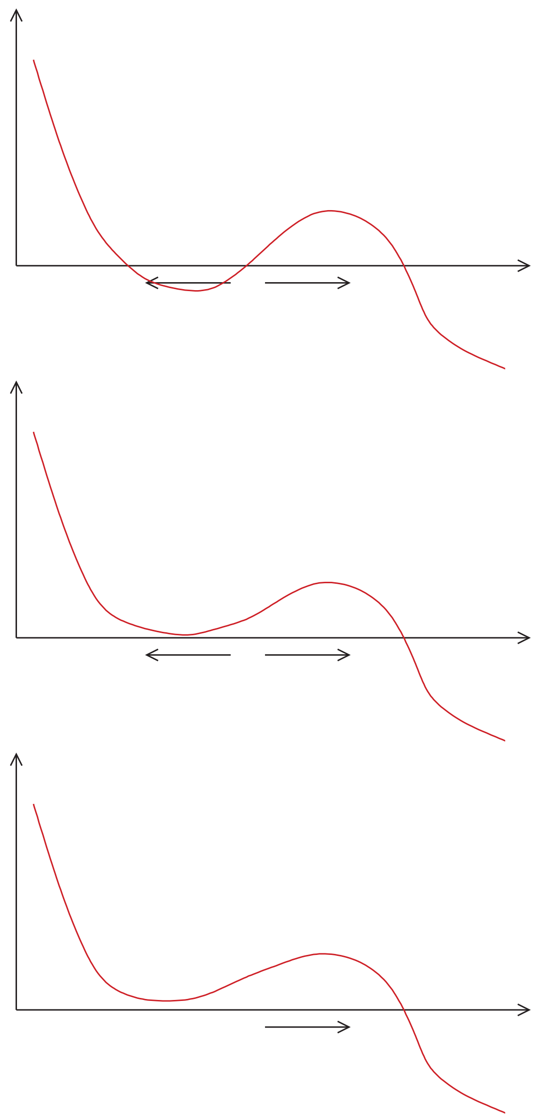
Black Monday

Intraday crashes that occur for no obvious reason like the Black Monday of October 19, 1987 have been explained by this switching from one equilibrium to another [Genotte and Leland, 1990].



Illustration

How can a such a switching occur?
Initially: 3 equilibria (2 stable).
Then: one becomes unstable.
For more details see book, Sec. 4.7.



Stability theorem

We state the stability result formally:

Theorem ("Law of supply and demand")

Assume that agents have mean-variance preferences of the form

$$U^i(c_1^i, \ldots, c_S^i) \;=\; \mu^i(c_1^i, \ldots, c_S^i) - \frac{\gamma^i}{2}\sigma^{2,i}(c_1^i, \ldots, c_S^i). \qquad (1)$$

Then there is a unique globally stable market equilibrium.

The proof can be found in the text book on page 205f.

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